WI3417TU
Kansmodellen voor Finance
Assignment 12

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December 20, 2011

Chapter 5, Exercise 3

(i)

$$
\begin{aligned}
pe^{\sigma} + qe^{-\sigma} &= 1 \\
p + q\left(e^{-\sigma}\right)^2 &= e^{-\sigma} \\
q\left(e^{-\sigma}\right)^2 - e^{-\sigma} + p &= 0 \\
e^{-\sigma} &= \frac{1+\sqrt{1-4pq}}{2p} \\
-\sigma &= \ln\left(\frac{1+\sqrt{1-4pq}}{2p}\right) \\
&= \ln(1+\sqrt{1-4pq}) - \ln(2p) \\
&= \ln(1+\sqrt{1-4p(1-p)}) - \ln(2p) \\
&= \ln(1+\sqrt{1-4p+4p^2}) - \ln(2p) \\
&= \ln(1+\sqrt{(1-2p)^2}) - \ln(2p) \\
&= \ln(1+(1-2p)) - \ln(2p) \\
&= \ln(2-2p) - \ln(2p) \\
\sigma &= \ln(2p) - \ln(2-2p) \\
\sigma &= \ln\left(\frac{2p}{2-2p}\right) \\
\sigma &= \ln\left(\frac{p}{q}\right)
\end{aligned}
$$

(ii)

We know from 5.2 that the following is a martingale:

$$S_n = e^{\sigma M_n} \left(\frac{1}{f(\sigma)} \right)^n$$
$$S_{n+1} = S_n e^{\sigma X_{n+1}} \left(\frac{1}{f(\sigma)} \right)$$

Taking the limit as n goes to infinity of:

$$1 = S_0 = \mathbb{E} S_{n \wedge \tau_1} = \mathbb{E} \left[e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} \right]$$

Gives:

$$\mathbb{E} \left[\mathbb{I}_{\{\tau_1 < \infty\}} e^{\sigma} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] = 1$$

Taking the limit of this as $\sigma \downarrow \sigma_0$ we find that:

$$\mathbb{P}\{t_1 < \infty\} = \frac{p}{q}$$

(iii)

If we set α to satisfy:

$$\alpha = \left(\frac{1}{f(\sigma)} \right)$$

Because of (i) we know that taking $\sigma > \sigma_0$, we get $\alpha \in (0, 1)$.
Solving for e^{σ} gives:

$$\alpha = \frac{1}{pe^{\sigma} + qe^{-\sigma}}$$
$$\alpha pe^{\sigma} + \alpha qe^{-\sigma} - 1 = 0$$
$$\alpha p \left(e^{\sigma}\right)^2 - e^{\sigma} + \alpha q = 0$$
$$e^{\sigma} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha p}$$

Filling in this gives us:

$$\mathbb{E}\left[\mathbb{I}_{\{\tau_1 < \infty\}} \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha p} \alpha^{\tau_1}\right] = 1$$
$$\mathbb{E}\left[\mathbb{I}_{\{\tau_1 < \infty\}} \alpha^{\tau_1}\right] = \frac{2\alpha p}{1 - \sqrt{1 - 4\alpha^2 pq}}$$
$$\mathbb{E}[\mathbb{I}_{\{\tau_1 = \infty\}} \alpha^{\tau_1}] = 0 \qquad \alpha < 1$$
$$\mathbb{E}\left[\alpha^{\tau_1}\right] = \frac{2\alpha p}{1 - \sqrt{1 - 4\alpha^2 pq}}$$

(iv)

Taking the derivative of:

$$\frac{\partial}{\partial \alpha} \mathbb{E}\left[\mathbb{I}_{\{\tau_1 < \infty\}} \alpha^{\tau_1}\right] = \frac{\partial}{\partial \alpha} \frac{2\alpha p}{1 - \sqrt{1 - 4\alpha^2 pq}}$$
$$\mathbb{E}\left[\mathbb{I}_{\{\tau_1 < \infty\}} \tau_1 \alpha^{\tau_1 - 1}\right] = \frac{\frac{1}{\sqrt{1 - 4\alpha^2 pq}} - 1}{2a^2 p}$$

Taking the limit of this as $\alpha \uparrow 1$ (that is, $\sigma \downarrow \sigma_0$):

$$\mathbb{E}\left[\mathbb{I}_{\{\tau_1 < \infty\}} \tau_1\right] = \frac{\frac{1}{\sqrt{1 - 4pq}} - 1}{2p}$$
$$\mathbb{E}\left[\mathbb{I}_{\{\tau_1 < \infty\}} \tau_1\right] = \frac{\frac{1}{1 - 2p} - 1}{2p}$$
$$\mathbb{E}\left[\mathbb{I}_{\{\tau_1 < \infty\}} \tau_1\right] = \frac{1}{2p - 4p^2} - \frac{1}{2p}$$

Chapter 5, Exercise 4

(ii)

$$\mathbb{P}(\tau_2 = 2k) = \mathbb{P}(\tau_2 \le 2k) - \mathbb{P}(\tau_2 \le 2k-2)$$

$$\begin{aligned}\mathbb{P}(\tau_2 \le 2k) &= \mathbb{P}(M_{2k} = 2) + 2\mathbb{P}(M_{2k} \ge 4) \\ &= \mathbb{P}(M_{2k} = 2) + \mathbb{P}(M_{2k} \ge 4) + \mathbb{P}(M_{2k} \le -4) \\ &= 1 - \mathbb{P}(M_{2k} = 0) - \mathbb{P}(M_{2k} = -2) \\ &= 1 - \frac{(2k)!}{k!k!}\left(\frac{1}{2}\right)^{2k} - \frac{(2k)!}{(k+1)!(k-1)!}\left(\frac{1}{2}\right)^{2k}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(\tau_2 = 2k) &= 1 - \frac{(2k)!}{k!k!}\left(\frac{1}{2}\right)^{2k} - \frac{(2k)!}{(k+1)!(k-1)!}\left(\frac{1}{2}\right)^{2k} - 1 + \frac{(2k-2)!}{(k-1)!(k-1)!}\left(\frac{1}{2}\right)^{2k-2} + \frac{(2k-2)!}{(k!(k-2)!}\left(\frac{1}{2}\right)^{2k-2} \\ &= -\left(\frac{1}{2}\right)^{2k}\left(\frac{(2k)!}{k!k!} + \frac{(2k)!}{(k+1)!(k-1)!}\right) + \left(\frac{1}{2}\right)^{2k-2}\left(\frac{(2k-2)!}{(k-1)!(k-1)!} + \frac{(2k-2)!}{k!(k-2)!}\right) \\ &= -\left(\frac{1}{2}\right)^{2k}\left(\frac{(2k)!}{k!k!} + \frac{(2k)!}{(k+1)!(k-1)!}\right) + \left(\frac{1}{2}\right)^{2k}\left(\frac{4(2k-2)!}{(k-1)!(k-1)!} + \frac{4(2k-2)!}{k!(k-2)!}\right) \\ &= -\left(\frac{1}{2}\right)^{2k}\left(\frac{(2k)!}{k!k!} + \frac{(2k)!}{(k+1)!(k-1)!} + \frac{4(2k-2)!}{(k-1)!(k-1)!} + \frac{4(2k-2)!}{k!(k-2)!}\right) \\ &= -\left(\frac{1}{2}\right)^{2k}\left(\frac{(2k)!}{k!k!} + \frac{(2k)!}{(k+1)!(k-1)!} + \frac{4(2k-2)!}{(k-1)!(k-1)!} + \frac{4(2k-2)!}{k!(k-2)!}\right) \\ &= \left(\frac{1}{2}\right)^{2k}\frac{(2k)!}{(k+1)!k!}\end{aligned}$$