# WI3417TU
# Kansmodellen voor Finance
# Assignment 10

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## Chapter 5, Exercise 2

### i

$$\begin{aligned}
f(\sigma) &= pe^{\sigma} + qe^{-\sigma} \\
f(\sigma) &= pe^{\sigma} + (1-p)e^{-\sigma} \\
f(\sigma) &= p(e^{\sigma} - e^{-\sigma}) + e^{-\sigma} \\
f(\sigma) &> 0.5(e^{\sigma} - e^{-\sigma}) + e^{-\sigma} \\
f(\sigma) &> 0.5e^{\sigma} - 0.5e^{-\sigma} + e^{-\sigma} \\
f(\sigma) &> 0.5e^{\sigma} + 0.5e^{-\sigma} \\
f(\sigma) &> \cosh(\sigma) \\
f(\sigma) &> 1
\end{aligned}$$

### ii

$$\begin{aligned}
S_{n+1} &= S_n e^{\sigma X_{n+1}} \left(\frac{1}{f(\sigma)}\right) \\
\mathbb{E}_n[S_{n+1}] &= S_n \left(\frac{1}{f(\sigma)}\right) \mathbb{E}_n[e^{\sigma X_{n+1}}] \\
&= S_n \left(\frac{1}{f(\sigma)}\right) \left(pe^{\sigma} + qe^{-\sigma}\right) \\
&= S_n
\end{aligned}$$

### iii

$S_n$ is a martingale, so $S_{n \wedge \tau_1}$ is as well. That is:

$$
\begin{aligned}
1 = S_0 = \mathbb{E}S_{n\wedge\tau_1} &= \mathbb{E}\left[e^{\sigma M_{n\wedge\tau_1}}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1}\right] \\
\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} &= e^{\sigma} && \text{if } \tau_1 < \infty \\
\lim_{n\to\infty}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} &= \left(\frac{1}{f(\sigma)}\right)^{\tau_1} && \text{if } \tau_1 < \infty \\
&= 0 && \text{if } \tau_1 = \infty
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} &= e^{\sigma}\left(\frac{1}{f(\sigma)}\right)^{\tau_1} && \text{if } \tau_1 < \infty \\
&= 0 && \text{if } \tau_1 = \infty
\end{aligned}
$$

We can take the limit from the martingale to get:

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{I}_{\{\tau_1<\infty\}} e^{\sigma}\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] &= 1 \\
\mathbb{E}\left[\mathbb{I}_{\{\tau_1<\infty\}}\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] &= \frac{1}{e^{\sigma}} = e^{-\sigma}
\end{aligned}
$$

Therefore, $P\{\tau_1 < \infty\} = 1$.

## iv

We solve for $\alpha \in (0,1), \sigma > 0$, which satisfies:

$$
\begin{aligned}
\alpha &= \frac{1}{f(\sigma)} \\
\alpha p e^{\sigma} + \alpha(1-p)e^{-\sigma} &= 1 \\
\alpha(1-p)(e^{-\sigma})^2 - e^{-\sigma} + \alpha p &= 0 \\
e^{-\sigma} &= \frac{-1 \pm \sqrt{1 - 4 * \alpha(1-p) * \alpha p}}{2\alpha(1-p)} \\
e^{-\sigma} &= \frac{-1 \pm \sqrt{1 - 4 * \alpha(1-p) * \alpha p}}{2\alpha(1-p)}
\end{aligned}
$$

From (iii) and having set $\alpha = \frac{1}{f(\sigma)}$, we can conclude that:

$$\mathbb{E}\alpha^{\tau_1} = \frac{-1 \pm \sqrt{1 - 4 * \alpha(1-p) * \alpha p}}{2\alpha(1-p)}$$

**V**

$$\begin{aligned}
\mathbb{E}\left[\tau_1 \alpha^{\tau_1 - 1}\right] &= \frac{\delta}{\delta\alpha} \mathbb{E}\alpha^{\tau_1} \\
&= \frac{\frac{1}{\sqrt{4a^2(p-1)p+1}} + 1}{2a^2(p-1)} \\
\lim_{\alpha \to 1} \mathbb{E}\left[\tau_1 \alpha^{\tau_1 - 1}\right] &= \mathbb{E}[\tau_1]
\end{aligned}$$